third-switching lines are to the right of this same straight line. It is sufficient to establish the diminution of the third function in (2.17) along (2.14) ($\nu = -1$) and the nonpositiveness of $\partial\psi_{13} / \partial l_2$. We have

$$\left.\frac{\partial\psi_{13}}{\partial\mu_{12}}\right|_{(2.14)} = (\tau_3 - l_2)(2 + l_2^2)\, l_2^{-1} \sin \tau_3 < 0$$

since $\tau_3 < l_2$ follows from the inequalities $\psi_{13} < 0,\ \mu_{13} + 1 > 0$. Further

$$\partial\psi_{13} / dl_2 = \mu_{22} [(\tau_3' - 1)\ (\mu_{12} + 1)\, \mu_{22}^{-1} + (\tau_3 - l_2)\ (\cos \tau_3 + \mu_{23}\mu_{22}^{-1}\tau_3'] \tag{5.2}$$

From (2.21)

$$\tau_3' = \tau_3 \sin^2 \tau_3\ (2l_2 - \tau_3)\ [(1 + l_2^2) \sin^2 \tau_3 + \tau_3^2]^{-1}$$

Obviously, $0 < \tau_3' < 1$. By Lemma 1, $\cos \tau_3 < 0$. Consequently, (5.2) is nonpositive.

REFERENCES

1. Krasovskii, N. N., Theory of Control of Motion. Linear Systems. Moscow, "Nauka", 1968.
2. Zavalishchin, S. T., Minimization of an integral square deviation by an impulse control. Avtomatika i Telemekhanika, №1, 1973.
3. Bellman, R., Dynamic Programing. Princeton, Univ. Press, 1957.
4. Pontriagin, L. S., Boltianskii, V. G., Gamkrelidze, R. V. and Mishchenko, E. F., Mathematical Theory of Optimal Processes. Moscow, Fizmatgiz, 1961. (See also English translation, Pergamon Press, Pontriagin, L. S., Book №10176, 1964).

Translated by N. H. C.

UDC 531.31

CERTAIN PROPERTIES OF PRECESSIONAL MOTIONS RELATIVE TO THE VERTICAL OF A HEAVY SOLID BODY WITH ONE FIXED POINT

PMM Vol. 38, №3, 1974, pp. 451-458

G. V. GORR
(Donetsk)

(Received July 6, 1973)

We prove new properties of the precessional motions relative to the vertical of a heavy solid body having a fixed point. In particular, we have shown that semi-regular precessions are possible only in the Hesse solution, while in the case when the precession rate and the self-rotation velocity are not constant, the constant of the integral of the angular momentum equals zero.

1. Statement of the problem. Definition [1-3]. The precessional motions of a solid body with one fixed point are the motions under which the angle between two straight lines, one of which is fixed in the body, while the other is fixed in a nonmoving space, remains constant.

Let $\mathbf{k}$ and $\mathbf{v}^*$ be unit vectors fixed, respectively, in the body and in space, and let ϑ be the angle between them. Then, the body's motion is a precession if $\vartheta = \text{const}$. By introducing into consideration the Euler angles $\vartheta,\ \varphi,\ \psi$, we obtain the expression

$$\omega = \varphi^{\cdot}\mathbf{k} + \psi^{\cdot}\mathbf{v}^* \tag{1.1}$$

for the angular velocity vector ω $(\omega_1, \omega_2, \omega_3)$, showing that the vector ω lies during the whole time of motion in the plane passing through the vectors $\mathbf{k}$ and $\mathbf{v}^*$. If in formula (1.1) $\varphi^{\cdot}$ and $\psi^{\cdot}$ are constants, then such motions of the body are called regular precessions. Motions in the case $\psi^{\cdot} = \text{const}$ are called semiregular precessions [2].

Appel'rot [3] studied precessions relative to the vertical ($\mathbf{v}^*$ coincides with the unit vector $\mathbf{v}$ in the direction of the force of gravity) of gyroscopes whose inertia ellipsoid at the fixed point is an ellipsoid of revolution. He showed that the precessions of such gyroscopes are impossible if the center of gravity does not lie on the perpendicular to the circular cross section of the inertia ellipsoid. Subsequently the papers of Italian scientists were devoted mainly to precessional motions. Grioli [1] examined regular precessions relative to a sloping axis (the vector $\mathbf{v}^*$ does not coincide with vector $\mathbf{v}$). Taking the additional condition $\varphi = \psi$, he found a new solution characterized by the conditions $\theta = \pi / 2$, $\omega_3 = \text{const}$. The center of gravity of Grioli's gyroscope lies on the perpendicular ($\mathbf{k}$) to the circular cross section of the inertia ellipsoid. Bressan [4], investigating precessional motions in the Hesse solution, showed that in this solution there exist only precessions relative to the vertical $\mathbf{v}$ and the horizontal axis $\mathbf{v}^*$ (the angle between vectors $\mathbf{v}$ and $\mathbf{v}^*$ equals $\pi / 2$). We note that the constant of the integral of the angular momentum is nonzero in the first case and equals zero in the second. The examples of precessional motions given by the Italian scientists in the problem of the motion of a heavy solid body having a fixed point are not exhaustive since an analysis of special solutions in this problem shows that, for example, Dokshevich's solution [5] describes a precessional motion relative to the vertical. Therefore, it is of interest to obtain the general properties of precessions and to point out all the cases of integrability which are characterized by a precessional motion.

Let us examine the precession of a heavy solid body relative to vertical $\mathbf{v}$. With the body we connect a right-handed coordinate system in such a way that the third coordinate axis passes through vector $\mathbf{k}$. By ν_1, ν_2, ν_3 we denote the components of unit vector $\mathbf{v}$ and by e_1, e_2, e_3 the components of the unit vector in the direction from the fixed point to the body's center of gravity. By a rotation of the moving coordinate system around the third axis we achieve the equality $e_2 = 0$, which does not restrict the problem's generality. Taking the Euler angles as the variables of the problem, from (1.1) we find ($\mathbf{v}^* = \mathbf{v}$)

$$\nu_1 = \sin\vartheta \sin\varphi, \quad \nu_2 = \sin\vartheta\cos\varphi, \quad \nu_3 = \cos\vartheta$$
$$\omega_1 = \psi^{\cdot}\sin\vartheta\sin\varphi, \quad \omega_2 = \psi^{\cdot}\sin\vartheta\cos\varphi, \quad \omega_3 = \psi^{\cdot}\cos\vartheta + \varphi^{\cdot}$$

Using the notation in [6] we write the equations of motion of the solid body in the case of precessions $\vartheta^{\cdot} = 0$ as

$$\begin{gathered}\psi^{\cdot\cdot}(a_1\sin\varphi + a_2\cos\varphi + a_3) + A_{13}\varphi^{\cdot\cdot} - A_{23}\varphi^{\cdot 2} + \psi^{\cdot}\varphi^{\cdot}\times \\ (a_4\cos\varphi - 2a_2\sin\varphi - 2a_5) + \psi^{\cdot 2}(a_6\cos 2\varphi + \\ a_7\sin 2\varphi + a_8\cos\varphi - a_9\sin\varphi + a_{10}) + a_{11}\cos\varphi = 0\end{gathered} \tag{1.2}$$

$$\begin{gathered}\psi^{\cdot\cdot}(b_1\cos\varphi + a_2\sin\varphi + a_5) + A_{23}\varphi^{\cdot\cdot} + A_{13}\varphi^{\cdot 2} + \\ \psi^{\cdot}\varphi^{\cdot}(2a_2\cos\varphi + b_2\sin\varphi + 2a_3) + \psi^{\cdot 2}(a_7\cos 2\varphi - \\ a_6\sin 2\varphi + a_9\cos\varphi + b_3\sin\varphi + b_4) - a_{11}\sin\varphi + b_5 = 0\end{gathered}$$

$$\psi^{\cdot\cdot}(c_0\cos\varphi + c_1\sin\varphi + c_2) + A_{33}\varphi^{\cdot\cdot} - (c_3\cos 2\varphi + c_4\sin 2\varphi + c_5\cos\varphi - c_6\sin\varphi)\psi^{\cdot 2} - c_7\cos\varphi = 0$$

Here a_i, b_i, c_i are constants. (Γ is the product of the body's weight and the distance from the fixed point to the center of gravity, A_{ij} are the components of the inertia tensor $\det \mathbf{A}$).

$$\begin{aligned}
a_1 &= A_{11}\sin\vartheta, \quad a_2 = A_{12}\sin\vartheta, \quad a_3 = A_{13}\cos\vartheta \\
a_4 &= (A_{11} - A_{22} + A_{33})\sin\vartheta, \quad a_5 = A_{23}\cos\vartheta, \\
a_6 &= {}^1/_2\, A_{23}\sin^2\vartheta \\
a_7 &= {}^1/_2\, A_{13}\sin^2\vartheta, \; a_8 = (A_{33} - A_{22})\sin\vartheta\cos\vartheta \\
a_9 &= A_{12}\sin\vartheta\cos\vartheta, \quad a_{10} = {}^1/_2\, A_{23}(\sin^2\vartheta - 2\cos^2\vartheta), \\
a_{11} &= e_3\Gamma\sin\vartheta \\
b_1 &= A_{22}\sin\vartheta, \quad b_2 = (A_{11} - A_{22} - A_{33})\sin\vartheta, \\
b_3 &= (A_{11} - A_{33})\sin\vartheta\cos\vartheta \\
b_4 &= {}^1/_2\, A_{13}(2\cos^2\vartheta - \sin^2\vartheta), \quad b_5 = e_1\Gamma\cos\vartheta \\
c_0 &= A_{23}\sin\vartheta, \quad c_1 = A_{13}\sin\vartheta, \quad c_2 = A_{33}\cos\vartheta, \\
c_3 &= A_{12}\sin^2\vartheta \\
c_4 &= {}^1/_2\,(A_{11} - A_{22})\sin^2\vartheta, \quad c_5 = A_{13}\sin\vartheta\cos\vartheta, \\
c_6 &= A_{23}\sin\vartheta\cos\vartheta, \quad c_7 = e_1\Gamma\sin\vartheta
\end{aligned} \tag{1.3}$$

The integrals of Eqs. (1.2) are

$$\varphi^{\cdot}\Phi_1 - \psi^{\cdot}\Phi_2 = k \tag{1.4}$$

$$A_{33}\varphi^{\cdot 2} + 2\varphi^{\cdot}\psi^{\cdot}\Phi_1 - \psi^{\cdot 2}\Phi_2 - 2(c_7\sin\varphi + E_1) = 0 \tag{1.5}$$

Here

$$\begin{aligned}
\Phi_1 &= c_0\cos\varphi + c_1\sin\varphi + c_2 \\
\Phi_2 &= c_4\cos 2\varphi - c_3\sin 2\varphi - 2c_5\sin\varphi - 2c_6\cos\varphi - c_8 \\
c_8 &= A_{33}\cos^2\vartheta + {}^1/_2\,(A_{11} + A_{22})\sin^2\vartheta, \quad E_1 = E + e_3\Gamma\cos\vartheta
\end{aligned} \tag{1.6}$$

By k and E we have denoted, respectively, the constants of the integrals of the angular momentum and the energies.

2. The cases $\varphi^{\cdot} = \text{const}$, $\psi^{\cdot} \neq \text{const}$ **and** $\varphi^{\cdot} \neq \text{const}$, $\psi^{\cdot} = \text{const}$. From integrals (1.4), (1.5) we find the dependence of $\psi^{\cdot}$ and $\varphi^{\cdot}$ on the variable φ

$$\psi^{\cdot} = \frac{1}{\Phi_2}(\Phi_1\varphi^{\cdot} - k) \tag{2.1}$$

$$\varphi^{\cdot 2} = \frac{1}{A_{33}\Phi_2 + \Phi_1^2}\left[2(c_7\sin\varphi + E_1)\Phi_2 + k^2\right] \tag{2.2}$$

We represent

$$\begin{aligned}
A_{33}\Phi_2 + \Phi_1^2 &= m_1\cos 2\varphi + m_2\sin 2\varphi + m_3 \\
m_1 &= {}^1/_2\,[A_{33}(A_{11} - A_{22}) + A_{23}^2 - A_{13}^2]\sin^2\vartheta \\
m_2 &= (A_{23}A_{13} - A_{33}A_{12})\sin^2\vartheta \\
m_3 &= {}^1/_2\,[A_{13}^2 + A_{23}^2 - A_{33}(A_{11} + A_{22})]\sin^2\vartheta
\end{aligned}$$

The denominators in (2.1), (2.2) do not vanish identically with respect to the variable

φ and, consequently, these formulas solve the integration problem for Eqs. (1.2) under the condition that they are consistent. To seek the sufficient conditions for the existence of precessions it is necessary to require that the values of $\varphi^{\cdot}$ and $\psi^{\cdot}$ in (2.1), (2.2) satisfy Eqs. (1.2). If we substitute them into the last equation in (1.2) we arrive at an identity. If we substitute (2.1), (2.2) into the equation obtained by the addition of the first and second equations in (1.2), multiplied, respectively, by $\sin\varphi$ and $\cos\varphi$, then once again we obtain an identity. Therefore, it suffices to substitute $\varphi^{\cdot}$ and $\psi^{\cdot}$ only in the first equation in (1.2).

Introducing $\psi^{\cdot}$ from (2.1) into the first equation in (1.2), we obtain

$$\begin{aligned}&\Phi_2\,[(a_1\sin\varphi + a_2\cos\varphi + a_3)\,\Phi_1 + A_{13}\Phi_2]\,\varphi^{\cdot\cdot} + \\ &\quad \varphi^{\cdot 2}\,\{(a_1\sin\varphi + a_2\cos\varphi + a_3)\,[\Phi_2\,(c_1\cos\varphi - c_0\sin\varphi) + \\ &\quad 2\Phi_1\Phi_3] - A_{23}\Phi_2^2 + \Phi_1\Phi_2\,(a_4\cos\varphi - 2a_2\sin\varphi - 2a_5) + \\ &\quad \Phi_1^2\,(a_6\cos 2\varphi + a_7\sin 2\varphi + a_8\cos\varphi - a_9\sin\varphi + a_{10})\} + \\ &\quad a_{11}\Phi_2^2\cos\varphi + k^2\,(a_6\cos 2\varphi + a_7\sin 2\varphi + a_8\cos\varphi - \\ &\quad a_9\sin\varphi + a_{10}) = k\varphi^{\cdot}\,[2\Phi_3\,(a_1\sin\varphi + a_2\cos\varphi + a_3) + \\ &\quad \Phi_2\,(a_4\cos\varphi - 2a_2\sin\varphi - 2a_5) + 2\Phi_1\,(a_6\cos 2\varphi + \\ &\quad a_7\sin 2\varphi + a_8\cos\varphi - a_9\sin\varphi + a_{10})]\end{aligned} \tag{2.3}$$

where

$$\Phi_3 = c_3\cos 2\varphi + c_4\sin 2\varphi + c_5\cos\varphi - c_6\sin\varphi$$

It is known that only a regular precession holds in the Lagrange gyroscope for constant $\varphi^{\cdot}$ and $\psi^{\cdot}$. Let us consider the case $\varphi^{\cdot} = \text{const}$, $\psi^{\cdot} \neq \text{const}$.

Theorem 1. Precessional motions of a heavy solid body relative to the vertical are dynamically impossible in the case $\varphi^{\cdot} = \text{const}$, $\psi^{\cdot} \neq \text{const}$.

Proof. Let us return to Eqs. (2.2), (2.3). Because $\varphi^{\cdot} = \text{const}$ these equations should be identities. By equating to zero the coefficients of $\sin 3\varphi$ and $\cos 3\varphi$ in expression (2.2) and the coefficients of $\sin 5\varphi$ and $\cos 5\varphi$ in Eq.(2.3), we find

$$e_1c_4 = 0,\quad e_1c_3 = 0,\quad e_3c_4 = 0,\quad e_3c_3 = 0.$$

Since $e_1^2 + e_3^2 = 1$, it is necessary to assume

$$c_3 = 0,\quad c_4 = 0 \tag{2.4}$$

or $A_{12} = 0$, $A_{11} = A_{22}$. Identifying Eq. (2.2) with respect to φ and taking (2.4) into account, we obtain

$$\begin{aligned}&m_1\varphi^{\cdot 2} - 2c_5c_7 = 0,\quad m_2\varphi^{\cdot 2} + 2c_6c_7 = 0,\quad E_1c_6 = 0\\ &2E_1c_5 + c_7c_8 = 0,\quad m_3\varphi^{\cdot 2} + (2E_1c_8 + 2c_5c_7 - k^2) = 0\end{aligned} \tag{2.5}$$

At first let $E_1 = 0$. From the fourth equation in (2.5) it follows that $c_7 = 0$ $(e_1 = 0)$. The first two equations in (2.5) yield the equalities $m_1 = 0$, $m_2 = 0$ and, consequently, from the relations for m_1, m_2 it follows that $A_{13} = 0$, $A_{23} = 0$. From formula (2.1) it follows that $\psi^{\cdot} = \text{const}$. Consequently, $E_1 \neq 0$, $c_6 = A_{23}\cos\vartheta = 0$. The assumption $\cos\vartheta = 0$ leads once again to the equalities $c_5 = 0$, $A_{13} = 0$, $A_{23} = 0$. Let $A_{23} = 0$, $\cos\vartheta \neq 0$. Equating to zero the coefficient of $\cos 4\varphi$ in Eq. (2.3), we arrive at the equality $A_{13} = 0$, whence once again it follows that $\psi^{\cdot} = \text{const}$. The theorem is proved.

Let us consider the case of a semiregular precession: $\psi^{\cdot} = \text{const}$.

Theorem 2. Semiregular precessions of a heavy solid body, having a fixed point relative to the vertical, take place only in the Hesse solution.

Proof. Assuming the precession rate to be constant, we substitute the value of $\varphi^{\cdot}$, found from (1.4), into Eqs. (1.2) and (1.5). The requirement that the relations obtained be identities with respect to the variable φ leads to the parameter conditions

$$e_1 = 0, \quad e_2 = 0, \quad e_3 = 1, \quad A_{12} = A_{23} = 0, \quad A_{13}^2 = A_{33}(A_{11} - A_{22}) \tag{2.6}$$

$$\psi^{\cdot 2} = \frac{\Gamma}{A_{22}\cos\vartheta}, \quad k = A_{22}\psi^{\cdot}\sin^2\vartheta, \quad E = \frac{1}{2}A_{22}\psi^{\cdot 2}\sin^2\vartheta - e_3\Gamma\cos\vartheta$$

The dependence of variable φ on time is established from (1.4)

$$\varphi^{\cdot} = -\frac{\psi^{\cdot}}{A_{33}}(A_{13}\sin\vartheta\sin\varphi + A_{33}\cos\vartheta)$$

The projection of the kinetic moment onto the third coordinate axis bearing the body's center of mass is $A_{13}\omega_1 + A_{33}\omega_3 = 0$. The linear invariant obtained and conditions (2.6) characterize a special case of Hesse's solution.

This theorem was proved for a gyrostat in [7].

3. The precession rate and the velocity of self-rotation are not constant. We now assume $\varphi^{\cdot} \neq \text{const}$, $\psi^{\cdot} \neq \text{const}$.

Theorem 3. The necessary condition for the existence of precessional motions of a heavy solid body, having a fixed point, in the case when the precession rate and the self-rotation velocity are not constant, is the equality $k = 0$.

Proof. Having set $\varphi^{\cdot}$ from (2.2) into relation (2.3), we obtain an equation of the following form:

$$\begin{aligned}(\alpha_9\cos 9\varphi + \alpha_9^*\sin 9\varphi + \ldots)^2 = \\ k^2(c_4c_7\sin 3\varphi + c_3c_7\cos 3\varphi + \ldots)\times \\ (m_1\cos 2\varphi + m_2\sin 2\varphi + m_3)^3(\beta_3\cos 3\varphi + \beta_3^*\sin 3\varphi + \ldots)^2\end{aligned} \tag{3.1}$$

Hence it follows that $\alpha_9 = 0$, $\alpha_9^* = 0$. Using (1.3), (1.6) we write out

$$\begin{aligned}e_1[(\varkappa_1 - 2\varkappa_4)(c_3m_2 + c_4m_1) + (\varkappa_2 + 2\varkappa_3)(c_3m_1 - c_4m_2)] + \\ 2e_3[(c_3m_2 + c_4m_1)^2 - (c_3m_1 - c_4m_2)^2] = 0\end{aligned} \tag{3.2}$$

$$\begin{aligned}e_1[(\varkappa_2 + 2\varkappa_3)(c_3m_2 + c_4m_1) - (\varkappa_1 - 2\varkappa_4)(c_3m_1 - c_4m_2)] - \\ 4e_3(c_3m_2 + c_4m_1)(c_3m_1 - c_4m_2) = 0\end{aligned}$$

Here

$$\begin{aligned}\varkappa_1 &= c_4(c_0a_2 - a_1c_1 + 2A_{13}c_4) + c_3(c_0a_1 + c_1a_2 - 2A_{13}c_3) \\ \varkappa_2 &= -c_3(c_0a_2 - a_1c_1 + 2A_{13}c_4) + c_4(c_0a_1 + c_1a_2 - 2A_{13}c_3) \\ \varkappa_3 &= (a_4 - a_1)(c_3c_1 + c_0c_4) - 2A_{23}(c_4^2 - c_3^2) + a_2(c_1c_4 - \\ &\quad c_0c_3) + a_6(c_0^2 - c_1^2) - 2a_7c_0c_1 \\ \varkappa_4 &= (a_4 - a_1)(c_1c_4 - c_0c_3) + 4A_{23}c_3c_4 - a_2(c_0c_4 + c_1c_3) + \\ &\quad a_7(c_0^2 - c_1^2) + 2a_6c_0c_1\end{aligned} \tag{3.3}$$

Because $e_1^2 + e_3^2 = 1$, the determinant of Eqs. (3.2) equals zero. This leads to the cases (1) $c_3 = 0$, $c_4 = 0$, (2) $m_1 = 0$, $m_2 = 0$

(3) $(\varkappa_1 - 2\varkappa_4)(c_3m_1 - c_4m_2) + (\varkappa_2 + 2\varkappa_3)(c_3m_2 + c_4m_1) = 0$

For Case (1) we have

$$A_{12} = 0, \ A_{11} = A_{22} \tag{3.4}$$

Under conditions (3.4) the equalities $\alpha_8 = 0$, $\alpha_8^* = 0$ can be written in the form

$$\begin{aligned} c_7 [\varkappa_3 (c_5 m_1 + c_6 m_2) - \varkappa_4 (c_5 m_2 - c_6 m_1)] &= 0 \\ c_7 [\varkappa_4 (c_5 m_1 + c_6 m_2) + \varkappa_3 (c_5 m_2 - c_6 m_1)] &= 0 \end{aligned} \tag{3.5}$$

If $\varkappa_3^2 + \varkappa_4^2 = 0$ or $m_1^2 + m_2^2 = 0$, then the third coordinate axis is the principal axis and, consequently, the precession becomes regular. Therefore, with due regard to notation (1.3), from (3.5) it follows that $e_1 \cos \vartheta = 0$. For $\cos \vartheta = 0$, Eq. (3.1) takes the form

$$[(-2 c_7 c_8 \sin \varphi + \ldots)(m_1 \cos 2\varphi + m_2 \sin 2\varphi + \ldots)(^1/_4 \varkappa_3 \times \cos 4\varphi + {}^1/_4 \varkappa_4 \sin 4\varphi + \ldots) + \ldots]^2 = k^2 (-2 c_7 c_8 \sin \varphi + \ldots)(m_1 \cos 2\varphi + m_2 \sin 2\varphi + \ldots)^3 [^1/_2 (A_{23}^2 - A_{13}^2) \sin^3 \vartheta \times \cos 3\varphi + \ldots]^2 \tag{3.6}$$

Here we have indicated only the terms with the largest multiple of angle φ. From (3.6), by equating the coefficient of $\sin 4\varphi$ to zero, we determine that $e_1 = 0$. Since under this condition the body's center of gravity lies on a perpendicular to a circular cross section of the inertia ellipsoid, by a rotation of the moving system around the third axis we achieve the equality $A_{23} = 0$. Returning to Eq. (3.1), we obtain

$$E_1 = 0, \ k = 0, \ \operatorname{tg}^2 \vartheta = A_{33}/A_{11}, \ A_{11} A_{33} - A_{13}^2 = 0$$

The last constraint cannot hold for a real motion since from the condition that the kinetic energy is a positive-definite quadratic form it follows that $A_{11} A_{33} - A_{13}^2 > 0$. Thus, only regular precessions are possible in the Case (1).

We look at Case (2). We have

$$A_{33} (A_{11} - A_{22}) + A_{23}^2 - A_{13}^2 = 0, \ A_{23} A_{13} - A_{33} A_{12} = 0 \tag{3.7}$$

If we assume $c_7 \neq 0$, then the equalities $\alpha_7 = 0$, $\alpha_7^* = 0$ lead to the equations

$$c_3 (2\varkappa_3 + 3\varkappa_2) - c_4 (2\varkappa_4 - 3\varkappa_1) = 0, \ c_3 (2\varkappa_4 - 3\varkappa_1) + c_4 \times (2\varkappa_3 + 3\varkappa_2) = 0 \tag{3.8}$$

Since the case $c_3^2 + c_4^2 = 0$ has been examined, from (3.8) follows:

$$3\varkappa_1 - 2\varkappa_4 = 0, \ 3\varkappa_2 + 2\varkappa_3 = 0$$

Substituting here $\varkappa_i$ $(i = 1, 2, 3, 4)$ from (3.3) and solving the equations obtained jointly with (3.7), we obtain

$$A_{11} = A_{12} A_{13} / A_{23}, \ A_{22} = A_{12} A_{23} / A_{13}, \ A_{33} = A_{23} A_{13} / A_{12}$$

Under these conditions the determinant $\det \mathbf{A}$ vanishes; this is mechanically meaningless. Therefore $e_1 = 0$. This condition shows that the body's center of gravity lies on a perpendicular to a circular cross section of the gyration ellipsoid and, consequently, the equality $A_{23} = 0$ can be achieved. By equating the coefficients of $\cos 8\varphi$ and $\cos 6\varphi$ in Eq. (3.1) to zero

$$E_1 = \frac{\Gamma \sin^2 \vartheta}{\cos \vartheta}, \quad A_{22}\,[\Gamma (A_{11} + A_{33}) \sin^4 \vartheta + k^2 \cos \vartheta]^2 = \qquad (3.9)$$
$$k^2 \Gamma \sin^4 \vartheta\, (A_{11} + A_{22} + A_{33})^2 \cos \vartheta$$

Solving Eq. (3.9) relative to k^2, we find two values for k^2

$$k^2_{(1)} = \frac{\Gamma (A_{11} + A_{33})^2}{A_{22} \cos \vartheta}, \qquad k^2_{(2)} = \frac{\Gamma A_{22} \sin^4 \vartheta}{\cos \vartheta}$$

Let $k = k_{(1)}$. Equating the coefficient of $\sin 5\varphi$ in Eq. (3.1) to zero leads to the condition $A_{11} - A_{22} + A_{33} = 0$, which contradicts (3.7). In the case $k = k_{(2)}$, from (2.1), (2.2) follows

$$\varphi^{\cdot} = \frac{-\Gamma \Phi_1}{k A_{33} \cos \vartheta}, \quad \psi^{\cdot} = \mathrm{const}$$

Consequently, condition (2) leads to the semiregular precession (case already considered).

Thus, condition (3) should be fulfilled when $\varphi^{\cdot} \neq \mathrm{const}$, $\psi^{\cdot} \neq \mathrm{const}$. Since for $e_1 = 0$ $(c_7 = 0)$ from Eqs. (3.2) we obtain either Case (1) or Case (2), then in what follows it is necessary to assume

$$c_7\,(m_1^2 + m_2^2)\,(c_3^2 + c_4^2) \neq 0 \qquad (3.10)$$

Let us now prove that $k = 0$. In fact, if we assume $k \neq 0$, then, by virtue of (3.10), from Eq. (3.1) it follows that $\beta_i = \beta_i^* = 0$ $(i = 0, 1, 2, 3)$. For $\cos \vartheta \neq 0$ we have

$$(A_{11} - A_{22})\,(A_{11} - A_{22} + A_{33}) - 2A_{23}^2 + 2A_{13}^2 = 0 \qquad (3.11)$$
$$A_{12}\,(A_{11} + A_{22} - A_{33}) + 2A_{13}A_{23} = 0, \; A_{23}\,(A_{11} - A_{22} - A_{33}) - 2A_{12}A_{13} = 0$$
$$A_{13}\,(A_{11} - A_{22} + A_{23}) + 2A_{12}A_{23} = 0$$

Solving Eqs. (3.11) relative to A_{11}, A_{22}, A_{23}, we obtain

$$A_{11} = -\frac{A_{23}\,(A_{12}^2 + A_{13}^2)}{A_{12} A_{13}}, \quad A_{22} = -\frac{A_{13}\,(A_{12}^2 + A_{23}^2)}{A_{12} A_{23}}$$
$$A_{33} = -\frac{A_{12}\,(A_{13}^2 + A_{23}^2)}{A_{13} A_{23}}$$

which leads to the equality $\det \mathbf{A} = 0$. If $\cos \vartheta = 0$, from the equalities $\beta_i = \beta_i^* = 0$ $(i = 0, 1, 2, 3)$ follow the first two equations in (3.11) and

$$A_{22}\,(A_{11} - A_{22} + A_{33}) - 2A_{12}^2 - 2A_{23}^2 = 0$$

Here once again $\det \mathbf{A} = 0$. Consequently, all the quantities β_i, β_i^* cannot vanish simultaneously; therefore, it is necessary to require $k = 0$ in Eq. (3.1).

We note the following property of precessions with nonconstant precession rates and self-rotation velocities: if $\varphi^{\cdot} \neq \mathrm{const}$, $\psi^{\cdot} \neq \mathrm{const}$, then the center of gravity does not lie on the straight line $(\boldsymbol{k})$ forming all through the motion a constant angle ϑ with the vertical. This property can be obtained from Eqs. (3.2). Setting $e_1 = 0$ in them, we arrive either to the Case (1) of regular precession or to the Case (2) of semiregular precession.

The author thanks P.V.Kharlamov for valuable advice while carrying out the work.

REFERENCES

1. Grioli, G., Esistenza e determinazione delle precessioni regolari dinamicamente possibili per un solido pesante asimmetrico. Ann. Mat. pura ed appl. Ser. IV, Vol. 26, № 3–4, 1947.
2. Grioli, G., Generalized precessions. Rev. Roumaine Sci. Techn. Ser. Mécanique Appliquée, Vol. 15, № 2, 1970.
3. Appel'rot, G. G., Not entirely symmetric heavy gyroscopes. In: Motion of a Solid Body Around a Fixed Point. Moscow-Leningrad, Izd. Akad. Nauk SSSR, 1940.
4. Bressan, A., On precessional motions of a solid body, relating to the Hesse case. Mekhanika, Sb. Perev., № 6, 1958.
5. Dokshevich, A. I., Integrable case of the problem of the motion of a heavy solid body around a fixed point. Prikl. Mekhan., Vol. 4, № 11, 1968.
6. Kharlamov, P. V., Lectures on Solid Body Dynamics, Pt. I. Novosibirsk, Izd. Novosibirsk. Univ., 1965.
7. Gorr, G. V., On one form of precessional motions of a gyrostat. In: Mechanics of a Solid Body, № 4, Kiev, "Naukova Dumka", 1972.

Translated by N. H. C.

UDC 531.31

RESONANCE MODES OF NEAR-CONSERVATIVE NONLINEAR SYSTEMS

PMM Vol. 38, № 3, 1974, pp. 459–464
Iu. V. MIKHLIN
(Dnepropetrovsk)
(Received July 10, 1973)

The Rauscher method is used to construct the steady-state resonance solutions of near-conservative nonautonomous multi-dimensional systems. It is assumed that the generating system has an analytic potential and admits of normal oscillations with rectilinear trajectories in configuration space. As is well known, the forced oscillations of systems with one degree of freedom in the resonance region are close to the natural oscillations of unperturbed conservative systems [1]. We present the possibility of generalizing this result to the multi-dimensional case, using the concept of normal forms of oscillations of conservative nonlinear systems [2, 3]. By selecting special types of external actions it was shown in [4] that the resonance modes possess the properties of the normal oscillations of conservative systems. For sufficiently general types of external periodic perturbations of quasi-linear systems close to Liapunov systems, Malkin [5] has exhaustively studied the periodic modes.

1. We consider the equations

$$x_s^{\cdot\cdot} = f_s(x_1, x_2, \ldots, x_n) + \varepsilon g_s(x_1, x_2, \ldots, x_n, t), \quad s = 1, 2, \ldots, n \qquad (1.1)$$

Here ε is a small parameter, f_s, g_s are analytic functions of $x_1, x_2, \ldots, x_n$; g_s is a periodic function of t of period T. We assume that the unperturbed system is conservative and admits of normal oscillations with rectilinear trajectories in configuration